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① Solve the integral equation

$$\phi(x) = e^x + \lambda\int_0^1 K(x,\xi)\,\phi(\xi)\,d\xi$$

where $K(x,\xi) = \begin{cases} \dfrac{-\sinh x \sinh(\xi-1)}{-\sinh 1}, & 0 \le x \le \xi \\ \dfrac{-\sinh \xi \sinh(x-1)}{-\sinh 1}, & \xi \le x \le 1 \end{cases}$

Solution: The given integral eqn is

$$\phi(x) = e^x + \lambda\int_0^1 K(x,\xi)\,\phi(\xi)\,d\xi \quad \text{——— (1)}$$

where

$$K(x,\xi) = \begin{cases} \dfrac{\sinh x \sinh(\xi-1)}{-\sinh 1}, & 0 \le x \le \xi \\ \dfrac{\sinh \xi \sinh(x-1)}{-\sinh 1}, & \xi \le x \le 1 \end{cases}$$

The homogeneous Integral Equation of (1) can be written as

$$\phi(x) = \lambda\int_0^x K(x,\xi)\,\phi(\xi)\,d\xi + \lambda\int_x^1 K(x,\xi)\,\phi(\xi)\,d\xi$$

$$\phi(x) = \lambda\int_0^x \frac{\sinh\xi \sinh(x-1)}{-\sinh 1}\phi(\xi)\,d\xi + \lambda\int_x^1 \frac{\sinh x \sinh(\xi-1)}{-\sinh 1}\phi(\xi)\,d\xi \quad \text{——— (2)}$$

Differentiating with respect to x, we have

$$\phi'(x) = \int_0^x \frac{\partial}{\partial x}\left\{\frac{\sinh\xi \sinh(x-1)}{-\sinh 1}\right\}\phi(\xi)\,d\xi$$

$$+ \lambda\int_x^1 \frac{\partial}{\partial x}\left\{\frac{\sinh x \sinh(\xi-1)}{-\sinh 1}\phi(\xi)\right\}d\xi$$

$$\Phi'(x) = \frac{\lambda \cosh(x-1)}{\sinh 1}\int_0^x \sinh\xi\, \Phi(\xi)\, d\xi$$

$$+ \left\{\frac{\sinh\xi \sinh(x-1)}{\sinh 1}\cdot\Phi(\xi)\right\}_{\xi=x}\frac{d}{dx}x - 0$$

$$+ \frac{\lambda\cosh x}{\sinh 1}\int_x^1 \sinh(\xi-1)\,\Phi(\xi)\, d\xi$$

$$+ \left\{\frac{\sinh x \sinh(\xi-1)}{\sinh 1}\Phi(\xi)\right\}_{\xi=1}\cdot\frac{d}{dx}\cdot 1$$

$$- \left\{\frac{\sinh x \sinh(\xi-1)}{\sinh 1}\Phi(\xi)\right\}_{\xi=x}\frac{d}{dx}x$$

$$\Phi'(x) = \frac{\lambda\cosh(x-1)}{\sinh 1}\int_0^x \sinh\xi\,\Phi(\xi)\, d\xi$$

$$+ \cancel{\frac{\sinh x \sinh(x-1)}{\sinh 1}}\Phi(x) + \frac{\lambda\cosh x}{\sinh 1}\int_x^1 \sinh(\xi-1)\,\Phi(\xi)\,d\xi$$

$$- \cancel{\frac{\sinh x\sinh(x-1)}{\sinh 1}}\Phi(x)$$

$$\Phi'(x) = \lambda\int_0^x \frac{\cosh(x-1)}{\sinh 1}\sinh\xi\cdot\Phi(\xi)\, d\xi$$

$$+ \lambda\int_x^1 \frac{\cosh x\,\sinh(\xi-1)}{\sinh 1}\Phi(\xi)\cdot d\xi \quad -(3)$$

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Again, differentiating, we have

$$\phi''(x) = \lambda \int_0^x \frac{\partial}{\partial x}\left\{\frac{\cosh(x-1)}{\sinh 1}\sinh\xi\,\phi(\xi)\right\} d\xi$$

$$+\lambda \int_x^1 \frac{\partial}{\partial x}\left\{\frac{\cosh x \sinh(\xi-1)}{\sinh 1}\cdot\phi(\xi)\right\} d\xi$$

$$= \lambda \int_0^x \frac{\sinh(x-1)}{\sinh 1}\sinh\xi\,\phi(\xi)\, d\xi$$

$$+\lambda\left\{\frac{\cosh(x-1)\sinh\xi}{\sinh 1}\phi(\xi)\right\}_{\xi=x}\frac{dx}{dx} - 0$$

$$+\lambda\int_x^1 \frac{\sinh x \sinh(\xi-1)}{\sinh 1}\phi(\xi)\,d\xi + 0 - \left\{\lambda\frac{\cosh x\sinh(\xi-1)}{\sinh 1}\phi(\xi)\right\}_{\xi=x}\cdot 1$$

$$\phi''(x) = \left[\lambda\int_0^x \frac{\sinh(x-1)\sinh\xi\,\phi(\xi)}{\sinh 1}d\xi + \lambda\int_x^1 \frac{\sinh x\sinh(\xi-1)}{\sinh 1}\phi(\xi)\,d\xi\right]$$

$$+\frac{\lambda\phi(x)}{\sinh 1}\left[\cosh(x-1)\sinh x - \cosh x\sinh(x-1)\right]$$

[$\because \sin(A-B) = \sin A\cos B - \cos A \sin B$]

$$\phi''(x) = \phi(x) + \frac{\lambda\phi(x)}{\cancel{\sinh 1}}\cancel{\sin[x-(x-1)]} \quad \text{, from (1)}$$

$$\Rightarrow \phi''(x) - (\lambda+1)\phi(x) = 0 \qquad \text{———— (4)}$$

The conditions are $\left.\begin{aligned}\phi(0)=0\\ \phi(1)=0\end{aligned}\right\}$ ———— (5)

Case I. If $\lambda+1=0$, Equation (4) $\Rightarrow$

$\phi''(x) = 0$ . . . 5.

$\Rightarrow \phi'(x) = A \Rightarrow \phi(x) = Ax + B$ —— (6)

$\phi(0) = 0 \Rightarrow 0 = B$

$\phi(1) = 0 \Rightarrow 1 = A + B \Rightarrow A = 0 \quad [\because B = 0]$

$\therefore$ (6) $\Rightarrow$

$\phi(x) = 0$ which is not an eigen function for $\lambda+1=0$ i.e for $\lambda = -1$.

Case II If $(\lambda+1) > 0$; Equation (4) $\Rightarrow$

$\phi''(x) - (\lambda+1)\phi(x) = 0$

$\Rightarrow$ A.E is $m^2 - (\lambda+1) = 0 \Rightarrow m = \pm\sqrt{\lambda+1}$

$\therefore \phi(x) = A\cosh(\sqrt{\lambda+1})x + B\sinh(\sqrt{\lambda+1})x$

$\phi(0) = 0$

$\Rightarrow 0 = A$

$\left\{ \cosh x = \dfrac{e^x + e^{-x}}{2}, \quad \sinh x = \dfrac{e^x - e^{-x}}{2} \right.$

$\phi(1) = 0 \Rightarrow$

$\phi(1) = A\cosh\sqrt{\lambda+1} + B\sinh\sqrt{\lambda+1}$

$\because \cosh 0 = 1, \sinh 0 = 0$

$\Rightarrow 0 = B\sinh\sqrt{\lambda+1} \quad [\because A = 0]$

$\Rightarrow B = 0$ as $\sinh\sqrt{\lambda+1} \neq 0$

$\therefore \phi(x) = 0$, which is not an eigen function

$\therefore (\lambda+1) > 0$, is not an eigen value.

Case III $(\lambda+1) < 0$ let $\lambda+1 = -\mu^2$ where $\mu > 0$

$\therefore$ eqn (4) $\Rightarrow$

$\phi'' - (-\mu^2)\phi = 0$

A.E $m^2 + \mu^2 = 0 \Rightarrow m = \pm \mu i$

Solⁿ is

$\phi(x) = A\cos\mu x + B\sin\mu x$ —— (6)

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$$\phi(0)=0 \Rightarrow A=0$$

$$\phi(1)=0 \Rightarrow 0 = B\sin\mu \Rightarrow \sin\mu = 0 \text{ if } B\neq 0$$

$$\left[\sin\mu = 0 = \sin n\pi \Rightarrow \mu = n\pi\right]$$

if $B=1$ then

$$\phi(x) = \sin\mu x \quad , \quad \mu = n\pi$$

$\therefore$ Required eigen fn is

$$\phi(x) = \sin\mu x \quad \text{where } 1+\lambda_n = -\mu^2$$

$$\boxed{\phi(x) = \sin n\pi x} \quad \Rightarrow \lambda_n = -1-\mu^2$$

$$\lambda_n = -1-n^2\pi^2 \quad \text{— (7)}$$

$$\Rightarrow \lambda_n = -(1+n^2\pi^2)$$

which is an eigen value.

Now
$$\bar{\phi}_n(x) = \frac{\phi_n(x)}{\|\phi_n(x)\|} = \frac{\sin n\pi x}{\|\sin n\pi x\|}$$

$$, \|\sin n\pi x\| = \sqrt{\int_0^1 \sin^2 n\pi x\, dx} = \sqrt{\int_0^1 \frac{1-\cos 2n\pi x}{2} dx}$$

$$= \sqrt{\int_0^1 \frac{1}{2} dx} = \sqrt{\frac{1}{2}}$$

$$\therefore \bar{\phi}_n(x) = \sqrt{2}\cdot \sin n\pi x \quad \text{— (8)}$$

$$F_n = \int_0^1 f(x)\cdot\bar{\phi}_n(x)\,dx = \int_0^1 e^x\cdot\sqrt{2}\sin n\pi x\, dx$$

$$= \sqrt{2}\cdot\left\{\frac{e^x}{1+n^2\pi^2}\left[\sin n\pi x - n\pi\cos n\pi x\right]\right\}_0^1$$

$$= \frac{\sqrt{2}}{1+n^2\pi^2}\left[e(0-n\pi\cos n\pi) - (0-n\pi)\right]$$

$$F_n = \frac{\sqrt{2}}{1+n^2\pi^2}\left[-n\pi e(-1)^n + n\pi\right] \quad \text{— (9)}$$

$\therefore$ Required soln is given by

$$\phi(x) = e^x + \sum_{n=1}^{\infty}\frac{\lambda}{\lambda_n-\lambda}F_n\bar{\phi}_n(x) \quad \text{for } \lambda\neq\lambda_n$$

$$\phi(x) = e^x + \sum_{n=1}^{\infty}\frac{\lambda}{-(1+n^2\pi^2)-\lambda}\cdot\frac{\sqrt{2}}{1+n^2\pi^2}\left[-n\pi e(-1)^n + n\pi\right] \times \frac{\sin n\pi x}{1}\sqrt{2}.$$

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(2) Find the eigen value and eigen functions of the homogeneous integral equation

$$\phi(x) = \lambda \int_1^2 \left(x\xi + \frac{1}{x\xi}\right)\phi(\xi)\,d\xi$$

Solution. Given homogeneous eqn is

$$\phi(x) = \lambda \int_1^2 \left(x\xi + \frac{1}{x\xi}\right)\phi(\xi)\,d\xi \qquad \text{—— (1)}$$

$$\phi(x) = \lambda x c_1 + \frac{\lambda}{x} c_2 \qquad \text{—— (2)}$$

where

$$c_1 = \int_1^2 \xi\phi(\xi)\,d\xi \qquad \text{—— (3)}$$

$$c_2 = \int_1^2 \frac{1}{\xi}\phi(\xi)\,d\xi \qquad \text{—— (4)}$$

From (2),

$$\phi(\xi) = \lambda c_1 \xi + \lambda c_2 \cdot \frac{1}{\xi} \qquad \text{—— (5)}$$

From (3) & (5)

$$c_1 = \int_1^2 \xi\left[\lambda c_1 \xi + \lambda c_2 \cdot \frac{1}{\xi}\right]d\xi$$

$$c_1 = \frac{\lambda c_1}{3}\left(\xi^3\right)_1^2 + \lambda c_2\left[\xi\right]_1^2 = \frac{\lambda c_1}{3}(7) + \lambda c_2[1]$$

$$\Rightarrow \left(1 - \frac{7\lambda}{3}\right)c_1 - \lambda c_2 = 0 \qquad \text{—— (6)}$$

From (4) and (5)

$$c_2 = \int_1^2 \frac{1}{\xi}\left[\lambda c_1 \xi + \lambda c_2 \cdot \frac{1}{\xi}\right]d\xi$$

$$= \lambda c_1\left[\xi\right]_1^2 + \lambda c_2\left[-\frac{1}{\xi}\right]_1^2 \qquad \int \frac{1}{\xi^2} = \frac{\xi^{-2+1}}{-2+1}$$

$$c_2 = \lambda c_1 - \lambda c_2\left[\frac{1}{2} - 1\right]$$

$$\Rightarrow -\lambda c_1 + \left(1 - \frac{\lambda}{2}\right)c_2 = 0 \qquad \text{—— (7)}$$

The $D(\lambda)$ for (6) & (7) is

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$$D(\lambda) = \begin{vmatrix} 1-\frac{7}{3}\lambda & -\lambda \\ -\lambda & 1-\frac{\lambda}{2} \end{vmatrix} = (1-\tfrac{7}{3}\lambda)(1-\tfrac{\lambda}{2}) - \lambda^2$$

$$= 1-\tfrac{7}{3}\lambda - \tfrac{\lambda}{2} + \tfrac{7}{3}\cdot\tfrac{1}{2}\lambda^2 - \lambda^2$$

To find eigen values, put $D(\lambda) = 0$

$$= \frac{6-14\lambda-3\lambda+7\lambda^2-6\lambda^2}{6}$$

$$= \left(\frac{\lambda^2-17\lambda+6}{6}\right)$$

$\therefore \quad \lambda^2 - 17\lambda + 6 = 0$

$$\lambda = \frac{+17 \pm \sqrt{17^2 - 4\times 6}}{2}$$

Taking positive sign, $\lambda_1 = 16.639$ (approximately)

Taking negative sign, $\lambda_2 = 0.3606$ (approximately)

To find eigen function corresponding to $\lambda_1 = 16.639$

We have $(1-\frac{7}{3}\lambda_1)c_1 - \lambda_1 c_2 = 0$

$$-\lambda_1 c_1 + (1-\tfrac{\lambda_1}{2})c_2 = 0$$

put $\lambda_1 = 16.639$

$$(1-\tfrac{7}{3}\times 16.639)c_1 - 16.639\, c_2 = 0$$

$$(1-38.824)c_1 = 16.639\, c_2$$

$$\Rightarrow c_2 = \frac{1-38.824}{16.639} c_1$$

putting the value of c_2 in

$$\phi(x) = \lambda_1\left[c_1 x + \frac{c_2}{x}\right]$$

$$\Rightarrow \phi_1(x) = \lambda_1\left[c_1 x + \frac{1}{x}\cdot\frac{1-38.824}{16.639}c_1\right]$$

$$\phi_1(x) = \lambda_1 c_1\left[x - 2.2732\cdot\frac{1}{x}\right]$$

$\Rightarrow \phi_1(x) = (x - 2.2732\cdot\frac{1}{x})$ if $\lambda_1 c_1 = 1$

which is required eigen fn corresponding to $\lambda_1 = 16.639$.

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To find eigen function corresponding to eigen value

$\lambda_2 = 0.3606$

We have

$$\left(1-\frac{7}{3}\lambda_2\right)c_1 - \lambda_2 c_2 = 0$$

$$-\lambda_2 c_1 + \left(1-\frac{\lambda_2}{2}\right)c_2 = 0$$

put $\lambda_2 = 0.3606$

$$\left(1-\frac{7}{3}0.3606\right)c_1 - 0.3606\, c_2$$

$$\Rightarrow c_1 = \frac{0.3606}{1-\frac{7}{3}(0.3606)}c_2$$

$$\Rightarrow c_1 = \frac{0.3606}{1-0.8414}c_2$$

$$\Rightarrow c_1 = \frac{0.3606}{0.1586}c_2$$

$$\Rightarrow c_1 = 2.2736\, c_2$$

$$\therefore \phi(x) = \lambda_2\left[c_1 x + \frac{c_2}{x}\right]$$

$$= \lambda_2\left[x\, 2.2736\, c_2 + \frac{c_2}{x}\right]$$

$$\phi(x) = 2.2736\,\lambda_2 c_2\left[x + \frac{1}{2.2736}\cdot\frac{1}{x}\right]$$

$$\Rightarrow \boxed{\phi_2(x) = x - 0.4398\cdot\frac{1}{x}} \quad , \because \text{if } 2.2736\,\lambda_2 c_2 = 1$$

which is required eigen function corresponding to the eigen value $\lambda_2 = 0.3606$.

(3) Find the eigen values and eigen function of

$$\phi(x) = \lambda \int_0^{\pi} K(x,\xi)\,\phi(\xi)\,d\xi \quad \text{where}$$

$$K(x,\xi) = \begin{cases} \cos x \sin\xi, & 0 \le x \le \xi \\ \cos\xi \sin x, & \xi \le x \le \pi \end{cases}$$

Solution. Given homogeneous integral equation is

$$\phi(x) = \lambda \int_0^{\pi} K(x,\xi)\,\phi(\xi)\,d\xi \quad \text{——— (1)}$$

$$K(x,\xi) = \begin{cases} \cos x \sin\xi, & 0 \le x \le \xi \\ \cos\xi \sin x, & \xi \le x \le \pi \end{cases}$$

Equ^n (1) $\Rightarrow$ $\phi(x) = \lambda \int_0^{x} K(x,\xi)\,\phi(\xi)\,d\xi + \lambda \int_x^{\pi} K(x,\xi)\,\phi(\xi)\,d\xi$

$$\phi(x) = \lambda \int_0^{x} \cos\xi \sin x\,\phi(\xi)\,d\xi + \lambda \int_x^{\pi} \cos x \sin\xi\,\phi(\xi)\,d\xi \quad \text{——— (2)}$$

Differentiating with respect to x, we have

$$\phi'(x) = \lambda \int_0^{x} \frac{\partial}{\partial x}\{\cos\xi \sin x\,\phi(\xi)\}\,.\,d\xi + \lambda \int_x^{\pi} \frac{\partial}{\partial x}\{\cos x \sin\xi\,\phi(\xi)\}\,d\xi$$

$$= \lambda \int_0^{x} \cos\xi \cos x\,\phi(\xi)\,d\xi + \{\cos\xi \sin x\,\phi(\xi)\}_{\xi=x} \cdot \frac{d}{dx}x - 0$$

$$+ \lambda \int_x^{\pi} -\sin x \sin\xi\,\phi(\xi)\,d\xi - \{\cos x \sin\xi\,\phi(\xi)\}_{\xi=x} \frac{dx}{dx} + 0$$

$$\phi'(x) = \lambda \int_0^{x} \cos\xi \cos x\,\phi(\xi)\,d\xi - \lambda \int_x^{\pi} \sin x \sin\xi\,\phi(\xi)\,d\xi \quad \text{——— (3)}$$

Again, differentiating, we have

$$\phi''(x) = \lambda \int_0^{x} \frac{\partial}{\partial x}\{\cos\xi \cos x\,\phi(\xi)\}\,d\xi + \cos^2 x\,\phi(x) - 0$$

$$-\left[\lambda \int_x^{\pi} \frac{\partial}{\partial x}\{\sin x \sin\xi\,\phi(\xi)\}\,d\xi + 0 - \sin^2 x\,\phi(x)\right]$$

$$\phi''(x) = -\lambda \int_0^x -\sin x \cdot \cos\xi\, \phi(\xi)\, d\xi - \lambda \int_x^{\pi} \cos x \sin\xi\, \phi(\xi)$$
$$+ \phi(x)$$

$\phi''(x) = -\lambda\, \phi(x) + \phi(x)$, from (2)

$\Rightarrow \phi''(x) - (\lambda-1)\phi(x) = 0$ ——— (4)

with the boundary condition

$$\left.\begin{array}{l} \phi(\pi)=0 \\ \phi'(0)=0 \end{array}\right\} \quad \text{——— (5)}$$

Case I when $(\lambda-1)=0$, equ (4) $\Rightarrow$

$\phi''(x)=0 \Rightarrow \phi(x) = Ax+B$

$\phi(\pi)=0 \Rightarrow 0 = A\pi + B$

$\phi'(x) = A$

$\phi'(0)=0 \Rightarrow A=0$

$\Rightarrow B=0$ $\left[\because \begin{array}{l} 0 = A\pi+B \\ 0 = 0\cdot\pi + B \end{array}\right]$

$\therefore \phi(x) = 0.x+0$

$\Rightarrow \phi(x)=0$, which is not an eigen function.

Case II if $(\lambda-1) > 0$ Let $\lambda - 1 = \mu^2 \Rightarrow \lambda = 1+\mu^2$

equ (4) $\Rightarrow$

$\phi''(x) - \mu^2\phi(x) = 0$

A.E is $m^2 - \mu^2 = 0 \Rightarrow m = \pm\mu$, $\mu \neq 0$

$\therefore \phi(x) = Ae^{-\mu x} + Be^{\mu x}$

$\phi(\pi)=0 \Rightarrow 0 = Ae^{-\mu\pi} + Be^{\mu\pi}$

$\phi'(x) = -\mu A e^{-\mu x} + B\mu e^{\mu x}$

$\phi'(0)=0 \Rightarrow 0 = -\mu A + \mu B \Rightarrow A = B$

$\therefore 0 = Ae^{-\mu\pi} + Ae^{\mu\pi}$ $[\because B=A]$

$\Rightarrow A=0$ as $e^{-\mu\pi} + e^{\mu\pi} \neq 0 \Rightarrow B=0$

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$\therefore$ $\phi(x) = 0 e^{-\mu x} + 0 e^{\mu x}$

$\Rightarrow \phi(x) = 0$, which is not an eigen function.

Case III when $\lambda - 1 < 0$ Let $\lambda - 1 = -\mu^2$, $\mu > 0$

$\Rightarrow \mu^2 = 1 - \lambda \Rightarrow \mu = \sqrt{1-\lambda}$

$\therefore$ Eqn (4) $\Rightarrow$

$\phi''(x) + \mu^2 \phi(x) = 0$

A.E is $m^2 + \mu^2 = 0 \Rightarrow m = \pm i\mu$

$\therefore$ $\phi(x) = A\cos\mu x + B\sin\mu x$, $\lambda = 1 - \mu^2$

$\phi(\pi) = 0 \Rightarrow 0 = A\cos\mu\pi + B\sin\mu\pi$ —— (6)

$\phi'(x) = -A\sin\mu x \cdot \mu + B\mu\cos\mu x$

$\phi'(0) = 0 \Rightarrow$ —— (7)

$0 = B\mu$

$D(\lambda)$ for (6) & (7) is

$$D(\lambda) = \begin{vmatrix} \cos\mu\pi & \sin\mu\pi \\ 0 & \mu \end{vmatrix} = \mu\cos\mu\pi$$

To find eigen values, put $D(\lambda) = 0$

$\Rightarrow \mu\cos\mu\pi = 0$

$\Rightarrow \cos\mu\pi = 0 = \cos \text{odd} \cdot \frac{\pi}{2}$

$\Rightarrow \mu\pi = (2n+1)\cdot\frac{\pi}{2} \Rightarrow \mu = n + \frac{1}{2}$

$\Rightarrow \sqrt{1-\lambda} = (n + \frac{1}{2}) \Rightarrow 1 - \lambda = (n+\frac{1}{2})^2$

$\Rightarrow \lambda = 1 - (n+\frac{1}{2})^2$

$\therefore \lambda_n = [1 - (n+\frac{1}{2})^2]$

Eqn (7) $\Rightarrow B\mu = 0 \Rightarrow B = 0$ as $\mu \neq 0$.

∴ Equ $\phi(x) = A\cos\mu x + B\sin\mu x$

$\Rightarrow \phi(x) = A\cos\mu x \qquad [\because B = 0]$

Take $A = 1$, $\mu = \sqrt{1-\lambda_n}$

$\phi_n(x) = \cos\{\sqrt{1-\lambda_n}\cdot x\}$

which is an eigen. fn corresponding to eigen value $\lambda_n = [1-(n+\frac{1}{2})^2]$.

(4) Using Hilbert schmidt theorem, solve

$$\phi(x) = x + \lambda\int_0^1 K(x,\xi)\phi(\xi)\,d\xi \quad \text{where}$$

$$K(x,\xi) = \begin{cases} x(\xi-1), & 0 \le x \le \xi \\ \xi(x-1), & \xi \le x \le 1 \end{cases}$$

Solution. Given

$$\phi(x) = x + \lambda\int_0^1 K(x,\xi)\phi(\xi)\,d\xi \qquad \text{—— (1)}$$

$$K(x,\xi) = \begin{cases} x(\xi-1), & 0 \le x \le \xi \\ \xi(x-1) & \xi \le x \le 1 \end{cases}$$

Consider the homogeneous equation of (1), we have

$$\phi(x) = \lambda\int_0^1 K(x,\xi)\phi(\xi)\,d\xi.$$

$$= \lambda\int_0^x K(x,\xi)\phi(\xi)\,d\xi + \lambda\int_x^1 K(x,\xi)\phi(\xi)\,d\xi$$

$$\phi(x) = \lambda\int_0^x \xi(x-1)\phi(\xi)\,d\xi + \lambda\int_x^1 x(\xi-1)\phi(\xi)\,d\xi \qquad \text{—— (2)}$$

Differentiating with respect to x, we have.

$$\phi'(x) = \lambda\int_0^x \frac{\partial}{\partial x}\left\{\xi(x-1)\phi(\xi)\right\} d\xi$$

$$+\lambda\int_x^1 \frac{\partial}{\partial x}\left\{x(\xi-1)\phi(\xi)\right\} d\xi$$

$$= \lambda\int_0^x \xi\phi(\xi)d\xi + \lambda\left\{\xi(x-1)\phi(\xi)\right\}_{\xi=x}\frac{dx}{dx} - 0$$

$$+\lambda\int_x^1 (\xi-1)\phi(\xi)d\xi + 0 - \lambda\left\{x(\xi-1)\phi(\xi)\right\}_{\xi=x}\frac{dx}{dx}$$

$$\phi'(x) = \lambda\int_0^x \xi\phi(\xi)d\xi + \cancel{\lambda x(x-1)\phi(x)} + \lambda\int_x^1(\xi-1)\phi(\xi)d\xi$$

$$\cancel{-\lambda x(x-1)\phi(x)}$$

$$\phi'(x) = \lambda\int_0^x \xi\phi(\xi)d\xi + \lambda\int_x^1 (\xi-1)\phi(\xi)d\xi$$

Again, differentiating we have

$$\phi''(x) = \lambda\int_0^x \frac{\partial}{\partial x}\left\{\xi\phi(\xi)\right\} d\xi + \lambda\int_x^1 \frac{\partial}{\partial x}\left\{(\xi-1)\phi(\xi)\right\} d\xi$$

$$= 0 + \lambda x\phi(x) - 0 + \lambda . 0 + 0 - \lambda(x-1)\phi(x)$$

$$\phi''(x) = \lambda x\phi(x) - \lambda x\phi(x) + \lambda\phi(x)$$

$$\phi''(x) - \lambda\phi(x) = 0 \quad \text{———} \quad (3)$$

with the boundary conditions

$$\phi(0) = 0 \text{ and } \phi(1) = 0 \quad \text{——} \quad (4)$$

Case I $\lambda = 0$, Eqn (3) $\Rightarrow$

$$\phi''(x) = 0 \Rightarrow \phi(x) = Ax + B$$

$$\phi(0) = 0 \Rightarrow 0 = A.0 + B \Rightarrow B = 0$$

$$\phi(1) = 0 \Rightarrow 0 = A.1 \Rightarrow A = 0$$

$\therefore$ $\phi(x) = 0.x + 0 = 0$, which is not an eigen function.

Case II $\lambda = +\mu^2$, $\mu > 0$ Then equ (3) $\Rightarrow$

$\phi''(x) - \mu^2\phi(x) = 0$

A.E is $m^2 - \mu^2 = 0 \Rightarrow m = \pm\mu$

$\phi(x) = Ae^{-\mu x} + Be^{\mu x}$

$\phi(0) = 0 \Rightarrow 0 = A + B \Rightarrow B = -A$

$\phi(1) = 0 \Rightarrow 0 = Ae^{-\mu} + Be^{\mu}$

$\Rightarrow 0 = Ae^{-\mu} - Ae^{\mu}$ $[\because B = -A]$

$\Rightarrow 0 = A[e^{-\mu} - e^{\mu}]$

$\Rightarrow A = 0$ as $(e^{-\mu} - e^{\mu}) \neq 0$

$\Rightarrow B = 0$ $[\because B = -A]$

$\therefore$ $\phi(x) = 0 \cdot e^{-\mu x} + 0 \cdot e^{\mu x}$

$\phi(x) = 0$, which is not an eigen function.

Case III $\lambda = -\mu^2$, $\mu > 0$ Then equ (3) $\Rightarrow$

$\phi''(x) + \mu^2\phi(x) = 0$

A.E is $m^2 + \mu^2 = 0 \Rightarrow m = \pm i\mu$

$\phi(x) = A\cos\mu x + B\sin\mu x$ where $-\mu^2 = \lambda$

$\phi(0) = 0 \Rightarrow 0 = A$

$\phi(1) = 0 \Rightarrow 0 = 0 + B\sin\mu$

if $B \neq 1$ then $\sin\mu = 0 = \sin n\pi$

$\Rightarrow \mu = n\pi$

$\therefore$ $\phi(x) = B\sin n\pi x$

if $B = 1$ then

$\phi_n(x) = \sin n\pi x$, $\|\phi_n(x)\| = \sqrt{\int_0^1 \sin^2 n\pi x\, dx}$

$= \sqrt{\int_0^1 \frac{1 - \cos 2n\pi x}{2} dx}$

$= \sqrt{\frac{1}{2}}$

$\therefore$ $\Phi(x) = \frac{\phi_n(x)}{\|\phi_n(x)\|} = \sqrt{2}\sin n\pi x$

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$$F_n = \int_0^1 f(x)\,\phi_n(x)\,dx$$

$$= \int_0^1 x \cdot \sqrt{2}\sin n\pi x\,dx$$

$$= \sqrt{2}\left[-\frac{x\cos n\pi x}{n\pi}\Big|_0^1 + \int_0^1 1 \cdot \frac{\cos n\pi x}{n\pi}\,dx\right]$$

$$= \sqrt{2}\left[-\frac{\cos n\pi}{n\pi} + \frac{1}{n\pi}\left(\sin n\pi x\right)_0^1\right]$$

$$F_n = -\frac{\sqrt{2}}{n\pi}(-1)^n$$

∴ Required solution is given by

$$\phi(x) = f(x) + \lambda\sum_{n=1}^{\infty}\frac{1}{\lambda_n-\lambda}F_n\,\phi_n(x)$$

$$= x + \frac{\lambda}{-n^2\pi^2-\lambda}\cdot\sqrt{2}(-1)^{n+1}\frac{\sin n\pi x\cdot\sqrt{2}}{n\pi}$$

$$\phi(x) = x + \frac{2\lambda}{\pi}\sum\frac{(-1)^n\sin n\pi x}{n(n^2\pi^2+\lambda)}$$

where $\lambda_n = -n^2\pi^2$, $n = 1, 2, 3, \ldots$

Q5. Using Hilbert Schmidt theorem solve

$$f(x) = \int_0^1 K(x,\xi)\,\phi(\xi)\,d\xi$$

$$\text{where } K(x,\xi) = \begin{cases} x(1-\xi), & x<\xi \\ \xi(1-x) & x>\xi \end{cases}$$

Take $f(x) = \lambda\int_0^x K(x,\xi)\,\phi(\xi)\,d\xi + \lambda\int_x^1 K(x,\xi)\,\phi(\xi)\,d\xi$

$$f(x) = \lambda\int_0^1 \xi(1-x)\,\phi(\xi)\,d\xi + \lambda\int_x^1 x(1-\xi)\,\phi(\xi)\,d\xi$$

& do as Q. No. 4. Here $\lambda = 1$, $\lambda_n = ?$